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Computer-assisted proof of lower bounds on the randomized query complexity of certain read-once threshold functions

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Abstract: **In this work we prove lower bounds on the randomized decision tree complexity of several read-once threshold functions. A read-once threshold formula can be defined by a rooted tree, in which every internal node is labeled by a threshold function T_k^n (with output 1 only when at least k out of n input bits are 1) and each leaf by a variable. We consider the randomized decision tree complexity of such functions, when the underlying tree is a uniform and balanced tree with all its internal nodes labeled by the same threshold function. We prove lower bounds of the form $c(k,n)^d$, where d is the depth of the tree, for certain values of k and n. The value of each parameter $c(k,n)$ depends on the solution of a linear program, which is provided by computational methods.**

Keywords: **Decision trees, query complexity, randomized computation, lower bounds, read-once functions.**

I. INTRODUCTION

One of the simplest computational models is the Boolean decision tree model and its randomized version. It is therefore interesting to study functions with complexity still unknown in these models. A notable example of such a function is the recursive majority-of-three function (see Example 1.2 in [11]), which has been studied in recent works [6, 7, 10, 8, 9, 3]. These papers have narrowed the gap between the upper and lower bounds for recursive majority. An $\Omega((7/3)^d)$ lower bound was showed in [6] using tools from information theory. Furthermore, they presented an algorithm that improves the $O((8/3)^d)$ upper bound shown in [11]. Magniez, Nayak, Santha, and Xiao [10], improved the lower bound to $\Omega((5/2)^d)$ and the upper bound to $O(2.64946^d)$. Subsequently, the lower bound was improved in [8] to $\Omega(2.55^d)$, building upon the techniques of [11]. The bound was further improved with a computer-assisted proof in [9]. The currently known best lower bound is $\Omega(2.59^d)$ from [3], where the proof was also computer-assisted.

The recursive majority-of-three function is contained in a more general class of functions, called *read-once threshold functions*. These can be represented by formulae involving threshold functions as connectives. A threshold k-out-of-n function, denoted T_k^n, is a Boolean function of n arguments that has value 1 if at least k of the n Boolean input values are 1. A threshold formula can be defined as a rooted tree with labeled nodes; each internal node is labeled by a threshold function and each leaf by a variable. If each variable appears exactly once the formula is called read-once. When there are no OR or AND gates, then the formula is non-degenerate (see Theorem 2.2 in [4]) and uniquely represents the corresponding function. Thus, we may define the depth of f, denoted $d(f)$, as the maximum depth of a leaf in the unique tree-representation and $n(f)$ as the number of variables.

Our work draws from the early work of Saks and Wigderson [11] that showed exact asymptotic bounds for nand_d, the function represented by a uniform binary tree with nand gates. Heiman and Wigderson [5] showed that for every read-once function f it holds $R(f) \in \Omega(D(f)^{0.51})$, where $R(f)$ and $D(f)$ are the randomized and deterministic complexity of f

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respectively. More recent work on this can be found in [1]. Threshold read-once functions were studied by Heiman, Newman, and Wigderson in [4] and were shown to have zero-error randomized complexity $\Omega(n/2^d)$.

In this work we provide lower bounds for those read-once threshold functions that are represented by uniform trees. That is, trees that are full and complete and with all their leaves at the same level. Furthermore, each internal node has the same number of children and is labeled by the same threshold gate T_k^n. We list lower bounds for several values of k and n and we prove them with computer-assisted calculations. To be precise, each lower bound depends on the optimal solution to a linear program. We note that our approach differs from [9, 3] and the linear program is different from the analogous computational problems encountered in these works.

Our results

Denote by $F_{k,n}^d$ the function represented by a uniform tree of depth d with each gate being T_k^n. With respect to these classes of read-once functions we prove lower bounds for several values of k and n on their randomized decision tree complexity, denoted $R(F_{k,n}^d)$.

Theorem 1 *For* $2 \le k \le 8$ *and* $3 \le n \le 9$, $R(F_{k,n}^d) = \Omega(c(k,n)^d)$, *where the values* $c(k,n)$ *are given in the following table.*

Table 1: Table of $c(k,n)$ values for $2 \le k \le 8$ and $3 \le n \le 9$.

n	k=2	k=3	k=4	k=5	k=6	k=7	k=8
3	2.500						
4	3.396	3.396					
5	4.359	4.125	4.359				
6	5.342	5.010	5.010	5.342			
7	6.332	5.945	5.812	5.945	6.332		
8	7.324	6.904	6.698	6.698	6.904	7.324	
9	8.320	7.881	7.622	7.535	7.622	7.881	8.320

The value of $c(2,3)$ matches the lower bound in [8], while the rest of the entries provide new lower bounds. More values for this table can be obtained with small computational power, however the size of the corresponding linear program grows exponentially with n.

We believe that this work might be useful in further research on the recursive majority-of-three function or on improving the bounds on general read-once threshold functions.

II. DEFINITIONS, NOTATION, PRELIMINARIES

Concepts related to decision tree complexity can be found in the survey of Buhrman and de Wolf [2].

Decision trees

A *deterministic Boolean decision tree* Q over a set of variables $Z = \{z_i 3mu|3mui \in [n]\}$, where $[n] = \{1,2,\dots,n\}$, is a rooted and ordered binary tree. Each internal node is labeled by a variable $z_i \in Z$ and each leaf with a value from $\{0,1\}$. An *assignment* to Z (or an *input* to Q) is a member of $\{0,1\}^n$. A $\sigma \in \{0,1,*\}^n$ is called a *cylinder*. The output $Q(\sigma)$ of Q on an input σ is defined recursively as follows. Start at the root and let its label be z_i. If $\sigma_i = 0$, we continue with the left child of the root; if $\sigma_i = 1$, we continue with the right child of the root. We continue recursively until we reach a leaf. We define $Q(\sigma)$ to be the label of that leaf. When we reach an internal node, we say that Q *queries* or *reads* or *probes* the corresponding variable. We say that Q *computes* a Boolean function $f: \{0,1\}^n \to \{0,1\}$, if for all $\sigma \in \{0,1\}^n$, $Q(\sigma) = f(\sigma)$. Note that every leaf of a decision tree determines a cylinder. The *cost of* Q *on input* σ, $cost\ (Q;\sigma)$, is the number of variables queried when the input is σ. The *cost of* Q, $cost\ (Q)$, is its *depth*, the maximum distance of a leaf from the root. The *deterministic complexity*, $D(f)$, of a Boolean function f is the minimum cost over all Boolean decision trees that compute f.

A *randomized Boolean decision tree* Q_R is a distribution p over deterministic decision trees. On input σ, a deterministic decision tree is chosen according to p and evaluated. The *cost of* Q_R *on input* σ is $cost\ (Q_R;\sigma) = \sum_Q p(Q) cost\ (Q;\sigma)$.

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The *cost* of Q_{R} is $\max_\sigma cost\ (Q_{\mathrm{R}};\sigma)$. A randomized decision tree Q_{R} *computes* a Boolean function f (with zero error), if $p(Q)>0$ only when Q computes f.

We are going to take a distributional view on randomized algorithms. Let μ be a distribution over $\{0,1\}^n$ and Q_{R} a randomized decision tree. The *expected cost of* Q_R *under* μ is $cost_\mu(Q_{\mathrm{R}})=\sum_\sigma \mu(\sigma)cost\ (Q_{\mathrm{R}};\sigma)$. The *expected complexity under* μ, $R_\mu(f)$, of a Boolean function f, is the minimum expected cost under μ of any randomized Boolean decision tree that computes f. Clearly, $R(f)\geq R_\mu(f)$, for any μ, and thus we can prove lower bounds on randomized complexity by providing lower bounds for the expected cost under any chosen distribution.

Trees representing functions

For a rooted tree T, the *depth* of a node is the number of edges on the path to the root; the *height* of a node is the number of edges on the longest path between the node and any descendant leaf. The *depth* of the tree is the maximum depth of a leaf. We call a tree *uniform* if all the leaves are on the same level and all internal nodes have the same number of children (i.e., it is full and complete). We denote by L_T the set of its leaves.

Consider the uniform tree of depth d in which every internal node has n children and is label by T_k^n (the k-out-of-n threshold gate) with $1<k<n$. We denote both the tree and the corresponding read-once threshold function by the same symbol $F_{k,n}^d$ as it will be clear from the context what it refers to. Furthermore, we might drop the subscripts k and n if they are secondary to our discussion.

Reluctant inputs and reluctant distribution

The inputs considered hard for such a function are the *reluctant* inputs ([11]). Let $M_{k,n}(0)=\binom{[n]}{k-1}$ and $M_{k,n}(1)=\binom{[n]}{k}$. Call an input to a T_k^n-gate *reluctant*, if it belongs to either of these sets. Call an input to a threshold read-once formula *reluctant*, if it is such that the inputs to every gate are reluctant. The *reluctant distribution* for a formula, is the uniform distribution over all reluctant inputs.

III. THE METHOD OF GENERALIZED COSTS

Our goal is to prove a lower bound on the expected cost of any randomized decision tree Q_{R} that computes a uniform threshold read-once function of depth d. The high-level outline of our proof is as follows. Given any decision tree Q_{R} for the function $F_{k,n}^d$, we define a randomized decision tree $Q_{\mathrm{R}}{}'$ that computes $F_{k,n}^{d-1}$. Algorithm $Q_{\mathrm{R}}{}'$ will use Q_{R} in a clever way so that

$$cost_\mu(Q_{\mathrm{R}})\geq\lambda\cdot cost_{\mu'}(Q_{\mathrm{R}}{}'),\quad where\ \lambda>1,\ \mu=\mu_{k,n}^d,\ \mu'=\mu_{k,n}^{d-1}.$$

Applying this step repeatedly we deduce that $cost_\mu(Q_{\mathrm{R}})$ is at least λ^d times the cost of an algorithm on a single variable.

To implement this plan we utilize the method of generalized costs of Saks and Wigderson [11]. We now recall some definitions from [8].

Cost functions

Define a *cost-function* relative to a variable set Z, to be a pair $c=(c_0,c_1)$, where $c_0,c_1\colon L_T\to\mathbb{R}$. The cost of a decision tree Q under cost-function c on input σ is

$$cost\ (Q;c;\sigma)=\sum_{z\in S} c_{\sigma_z}(z),$$

where $S=\{z3mu|3mu\ z\ is\ queried\ by\ Q\ on\ input\ \sigma\}$. The cost of a randomized decision tree Q_{R} on input σ under cost-function c is

$$cost\ (Q_{\mathrm{R}};c;\sigma)=\sum_Q p(Q)cost\ (Q;c;\sigma),$$

where p is the corresponding distribution over deterministic decision trees. Finally, the expected cost of a randomized decision tree Q_{R} under cost-function c and distribution μ is

$$cost_\mu(Q_{\mathrm{R}};c)=\sum_\sigma \mu(\sigma)cost\ (Q_{\mathrm{R}};c;\sigma).$$

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Preliminaries

Consider a tree T of depth $d+1$ such that all internal nodes have degree n and all leaves are on levels d and $d+1$. If we treat every internal node as a T_k^n-gate, this tree represents a function F and has an associated reluctant distribution μ. Suppose Q is a randomized decision tree that computes F and $c=(c_0,c_1)$ a cost function. We define a process that shrinks T to a smaller tree T' (of depth d or $d+1$) and also a corresponding randomized decision tree Q' that computes the function F' that is represented by T'. The crucial part is to show that for a "more expensive" cost-function c' and the reluctant distribution μ' over the variables of F',

$$cost_{\mu}(Q;c) \geq cost_{\mu'}(Q';c'). \qquad (1)$$

The main ingredient in this framework is the shrinking process, which entails removing n leaves $V=\{v_1,v_2,\dots,v_n\}$, so that their parent u would become a leaf in T'. Given an algorithm Q for F we obtain an algorithm Q' for F' as follows. On input $\sigma' u$, Q' first chooses uniformly at random $x \in [n-1]_{k-1}$ and $i \in [n]$ and then simulates Q on $\sigma = \sigma' x_1 \cdots x_{i-1} u x_i \cdots x_{n-1}$.

Fact 2 *If Q is a randomized decision tree that computes F, then Q' is a randomized decision tree that computes F'.*

Our goal is to determine the "most expensive" cost function c' for T' for which we can argue (1). To that end, it will be useful to express $cost_{\mu'}(Q';c')$ in terms of Q and T. For an assignment σ and a leaf w, let $b=T_k^n(\sigma_{v_1},\dots,\sigma_{v_n})$ and define a cost function $c=(c_0,c_1)$ for T as follows.

$$c_{\sigma_w} = \begin{cases} c'_{\sigma_w}(w) \ \ if \ \mathrm{w} \notin \mathrm{V} \\ c'_1(u)/k \ if \ \mathrm{w} \in \mathrm{V} \text{ and } \sigma_{\mathrm{w}} = \mathrm{b} = 1 \\ c'_0(u)/(n-k+1) \ if \ \mathrm{w} \in \mathrm{V} \text{ and } \sigma_{\mathrm{w}} = \mathrm{b} = 0 \end{cases} \qquad (2)$$

and 0 otherwise. For μ the reluctant distribution on T we have the following proposition.

Proposition 3 $cost_{\mu'}(Q';c') = cost_{\mu}(Q;c)$.

Proof. Observe that the distribution of σ as generated by Q' is μ. Furthermore, over the random choices of Q', each σ is encountered k and $n-k+1$ times when σ'_u is 1 or 0 respectively. In particular, for any σ, each time we charge Q for $c'_1(u)/k$, Q' is charged $c'_1(u)$ with probability $1/k$.

We shall define c' so that $c'(w)=c(w)$ for all $w \notin V$. Recall Proposition 3 and observe that

$$cost_{\mu}(Q;c) \geq cost_{\mu'}(Q';c') \Leftarrow cost_{\mu}(Q;c-\psi) \geq 0. \qquad (3)$$

Since $c-\psi$ is 0 for $w \notin V$, we only need to verify (3) with respect to the leaves in V. Thus, we are led to study decision trees over $\binom{[n]}{k}$.

Decision trees over $\binom{[n]}{k}$ and the quantity $P(k,n)$

We are interested in the cost of arbitrary (but non-empty) decision trees over $\mathcal{X}_{k,n} = \binom{[n]}{k}$. under the cost function $c_\eta = (1,-\eta)$, where $\eta \in \mathbb{R}_{\geq}$. In particular, we are interested in the values defined below.

Let $\mathcal{X}_{k,n} = [n]_k$ and $\nu_{k,n}$ the uniform distribution over $\mathcal{X}_{k,n}$. For $\eta \in \mathbb{R}_{\geq}$ and $0<k\leq n$, define

$$P_\eta(k,n) = \min_Q\{cost_{\nu_{k,n}}(Q;c_\eta))\} \quad and \quad P(k,n) = \max_\eta\{P_\eta(k,n) \geq 0\},$$

where Q ranges over all non-empty decision trees. Define also $P(0,n)=\infty$.

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Computing $P(k,n)$ is our main goal. Suppose $c_1(v_1) = \cdots = c_1(v_n) = c_1$, $c_0(v_1) = \cdots = c_0(v_n) = c_0$, $\alpha = P(k,n)$, and $\beta = P(n-k+1,n)$. Writing $c'_1 = c'_1(u)$ and $c'_0 = c'_0(u)$, we may set

$$(c_{1'}c_0) \quad = \begin{pmatrix} k & \alpha k \\ \beta(n-k+1) & n-k+1 \end{pmatrix} (c_1 c_0). \qquad (4)$$

Proposition 4 $cost_\mu(Q;c) \geq cost_{\mu'}(Q';c')$.

Proof. As discussed above, $c-\psi$ is 0 outside V. The inequality follows from the observation that $(c-\psi)$ is $c_0 c_\alpha = (c_0, -c_0\alpha)$ over the support of $\nu_{k,n}$ and $c_1 c_\beta = (c_1, -c_1\beta)$ over the support of $\nu_{n-k-1,n}$.

Applying this process repeatedly, shrinking all sibling leaves to their parent, we reduce F^d to F^{d-1}. After d such steps we are left with a single node and the cost function defined by

$$(c_{1'}c_0) \quad = \Gamma_{k,n}^d(1,1) \quad , \text{ where } \Gamma_{k,n} = \begin{pmatrix} k & \alpha k \\ \beta(n-k+1) & n-k+1 \end{pmatrix}. \quad (5)$$

We obtain a lower bound in the order of λ^d, where λ is the largest eigenvalue of the matrix $\Gamma_{k,n}$. Denoting its trace by T and its determinant by D,

$$\lambda = \frac{T}{2} + \sqrt{\frac{T^2}{4} - D}, \quad \text{where } T = n+1 \text{ and } D = (1-\alpha\beta)k(n-k+1). \qquad (6)$$

To obtain the best possible value for $c(k,n)$ (with respect to Theorem 1), we are now concerned with the task of providing the best possible lower bounds on $\alpha = P(k,n)$ and $\beta = P(n-k+1,n)$.

IV. A LINEAR PROGRAMMING APPROACH

We can lower bound $P(k,n)$ by the optimal value of a linear program. Note that a decision tree algorithm over $X = \mathcal{X}_{k,n}$ determines a subset of

$$Z = \{z \in \{0,1\}^m : 1 \leq m \leq n\}.$$

Denoting by n_z the length of z and by k_z its weight (the number of 1's), the cost of a $z \in Z$ under c_η can be written as

$$(n_z - k_z - \eta k_z) \cdot \frac{\binom{n-n_z}{k-k_z}}{\binom{n}{k}}.$$

The following linear program provides a lower bound on $P_\eta(k,n)$; i.e.a lower bound on the cost of any decision tree algorithm over X under c_η.

$$\min \quad \textstyle\sum_{z\in Z} \alpha_z (n_z - k_z - \eta k_z) \cdot \frac{\binom{n-n_z}{k-k_z}}{\binom{n}{k}}$$

$$\text{s.t.} \quad \alpha_z \geq 0, \forall z \in Z;$$

$$\textstyle\sum_{x\in z} \alpha_z = 1, \forall x \in X.$$

Where $x \in z$ denotes that z is a prefix of x (that is, x belongs in the cylinder defined by z). We obtain the following dual program.

$$\max \quad \textstyle\sum_{x\in X} \beta_x$$

$$\text{s.t.} \quad \textstyle\sum_{x\in z} \beta_x \leq (n_z - k_z - \eta k_z) \cdot \frac{\binom{n-n_z}{k-k_z}}{\binom{n}{k}}, \forall z \in Z.$$

Recall that we interested in the greatest value of η so that the value of the dual is non-negative. Observing that η is also a linear parameter, we manipulate the dual to obtain the following linear program. Normalizing by $|X| = n_k$,

$$\max \quad \eta$$

$$\text{s.t.} \quad \sum_{x\in X} \beta_x \geq 0$$

$$\eta k_z \binom{n-n_z}{k-k_z} + \sum_{x\in z} \beta_x \leq (n_z - k_z) \binom{n-n_z}{k-k_z}, \forall z \in Z.$$

We can now construct the dual of $D(k,n)$ to obtain a primal linear program for $P(k,n)$.

$$\min \quad \sum_{z\in Z} \alpha_z (n_z - k_z) \binom{n-n_z}{k-k_z}$$

$$\text{s.t.} \quad \sum_z k_z \binom{n-n_z}{k-k_z} \alpha_z = 1$$

$$\sum_{x\in z} \alpha_z = \alpha_\emptyset, \forall x \in X.$$

An alternative way to arrive at $LP(k,n)$ would be to express the objective function of the primal linear program $LP_\eta(k,n)$ as a ratio of linear functions and producing an equivalent linear program using standard techniques.

Lower bounds

We list in the following table numerical values of $P(k,n)$ that were computed by solving optimally the linear program $LP(k,n)$.

Table 2: Table of $P(k,n)$ values for $1 \leq k \leq 8$ and $2 \leq n \leq 9$.

n	k=1	k=2	k=3	k=4	k=5	k=6	k=7	k=8
2	0.5							
3	1	0.250						
4	1.5	0.555	0.166					
5	2	0.846	0.375	0.125				
6	2.5	1.142	0.594	0.285	0.100			
7	3	1.444	0.807	0.453	0.230	0.083		
8	3.5	1.736	1.021	0.624	0.367	0.192	0.071	
9	4	2.039	1.234	0.795	0.507	0.308	0.166	0.062

To obtain the corresponding table for Theorem 1 we apply Equation 6. For example, for $k = 3$ and $n = 7$, with respect to Equation 6 we obtain $T = 8$ and $D = 15 \cdot [1 - P(3,7)P(5,7)] = 15 \cdot (1 - 0.807 \cdot 0.230) = 12.215$, which gives $c(3,7) = 4 + \sqrt{16 - 12.215} = 5.945$.

V. CONCLUSIONS

We can lower We have shown how to generalize the framework in [8] to extend the lower bound that was obtained there for recursive majority-of-three to other threshold read-once functions. Further work could focus either on improving the bounds on recursive majority-of-three by formulating and solving a corresponding optimization problem or augmenting the analysis so that it includes any threshold read-once function and improving on the bound in [4].

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REFERENCES

[1] Kazuyuki Amano. Bounding the randomized decision tree complexity of read-once boolean functions. In *Proceedings of the Twenty-Second Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1729–1744. SIAM, 2011.

[2] Harry Buhrman and Ronald de Wolf. Complexity measures and decision tree complexity: a survey. *Theor. Comput. Sci.*, 288(1):21–43, 2002.

[3] Mika Göös and T. S. Jayram. A Composition Theorem for Conical Juntas. In Ran Raz, editor, *31st Conference on Computational Complexity (CCC 2016)*, volume 50 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 5:1–5:16, Dagstuhl, Germany, 2016. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

[4] Rafi Heiman, Ilan Newman, and Avi Wigderson. On read-once threshold formulae and their randomized decision tree complexity. *Theor. Comput. Sci.*, 107(1):63–76, 1993.

[5] Rafi Heiman and Avi Wigderson. Randomized vs. deterministic decision tree complexity for read-once boolean functions. *Computational Complexity*, 1:311–329, 1991.

[6] T. S. Jayram, Ravi Kumar, and D. Sivakumar. Two applications of information complexity. In *STOC*, pages 673–682. ACM, 2003.

[7] Itamar Landau, Asaf Nachmias, Yuval Peres, and Sithparran Vanniasegaram. The lower bound for evaluating a recursive ternary majority function and an entropy-free proof. *Undergraduate Research Reports, Department of Statistics, University of California, Berkeley*, 2006.

[8] Nikos Leonardos. An improved lower bound for the randomized decision tree complexity of recursive majority,. In Fedor V. Fomin, Rūsiņš Freivalds, Marta Kwiatkowska, and David Peleg, editors, *Automata, Languages, and Programming*, pages 696–708, Berlin, Heidelberg, 2013. Springer Berlin Heidelberg.

[9] Frédéric Magniez, Ashwin Nayak, Miklos Santha, Jonah Sherman, GΓ˝bor Tardos, and David Xiao. Improved bounds for the randomized decision tree complexity of recursive majority. *Random Structures & Algorithms*, 48(3):612–638, 2016.

[10] Frédéric Magniez, Ashwin Nayak, Miklos Santha, and David Xiao. Improved bounds for the randomized decision tree complexity of recursive majority. In Luca Aceto, Monika Henzinger, and Jiri Sgall, editors, *ICALP (1)*, volume 6755 of *Lecture Notes in Computer Science*, pages 317–329. Springer, 2011.

[11] Michael E. Saks and Avi Wigderson. Probabilistic boolean decision trees and the complexity of evaluating game trees. In *FOCS*, pages 29–38. IEEE Computer Society, 1986.

[12] V. P. Gountis and A. G. Bakirtzis, “Bidding strategies for electricity producers in a competitive electricity marketplace,” IEEE Trans. Power System, vol. 19, no. 1, pp. 356–365, Feb. 2004.

[13] J. Clerk Maxwell, “A Treatise on Electricity and Magnetism”, 3rd ed., vol. 2. Oxford: Clarendon, 1892, pp.68–73.

[14] R. Benato and A. Paolucci, EHV AC Undergrounding Electrical Power. Performance and Planning. New York: Springer, 2010.

[15] Angus DC, Linde-Zwirble WT, Lidicker J et al (2001) Epidemiology of severe sepsis in the United States: analysis of incidence, outcome, and associated costs of care. Crit Care Med 29:1303–1310.

[16] Levy MM, Dellinger RP, Townsend SR, Surviving Sepsis Campaign et al (2010) The Surviving Sepsis Campaign: results of an international guideline-based performance improvement program targeting severe sepsis. Crit Care Med 38:367–374.

[17] CIGRÉ Tech. Brochure # 379, “Update of service experience of HV underground and submarine cable systems,” 2009.

[18] E. E. Reber, R. L. Mitchell, and C. J. Carter, "Oxygen absorption in the Earth's atmosphere," Aerospace Corp., Los Angeles, CA, Tech. Rep. TR-0200 (4230-46)-3, Nov. 1968.